On the Logic of Classes as Many

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Abstract

The notion of a "class as many" was central to Bertrand Russell's early form of logicism in his 1903 *Principles of Mathematics*. There is no empty class in this sense, and the singleton of an urelement (or atom in our reconstruction) is identical with that urelement. Also, classes with more than one member are merely pluralities—or what are sometimes called "plural objects"—and cannot as such be themselves members of classes. Russell did not formally develop this notion of a class but used it only informally. In what follows, we give a formal, logical reconstruction of the logic of classes as many as pluralities (or plural objects) within a fragment of the framework of conceptual realism. We also take groups to be classes as many with two or more members and show how groups provide a semantics for plural quantifier phrases.

There is more than one notion of a class that has been described in the history of philosophy, and even the iterative concept of a set, which is the best known development, is not univocal. The iterative concept of set can be developed with an axiom of *anti*-foundation, for example, as well as with an axiom of foundation.[1] But in either case there can be no universal set, and yet there are set theories, such as NF and NFU, in which there is a universal set. Of course, the more traditional notions of a class were not based on the iterative concept of set, but on classes as the extensions of concepts (*Begriffsumfangen*), and in fact this may be the best way to understand NF and NFU.[2] Classes were understood in this way in Frege's *Grundgetsetze*, for example; and although Frege's theory was subject to Russell's paradox, it can

[1] For a development of set theory with an anti-foundation axiom, see Aczel 1988.

[2] See Cocchiarella 1987, Chapter 4, for a defense of this view. For a description and development of NFU, see Holmes 1998.

be reconstructed in a consistent manner.[3] Russell's type theory, of course, was based on a "no-classes" doctrine, according to which all talk of classes was to be explained away in terms of propositional functions, and hence it did not really contain a theory of classes. Still, one can ignore Russell's no-classes doctrine and develop a type-theoretical version of set theory instead.

There is a notion of a class that has been ignored by most, but not all, philosophers.[4] This is the notion of a "class as many," as described, e.g., by Bertrand Russell in his 1903 *Principles of Mathematics*.[5] A class in this sense is the extension of a common count noun, i.e., the extension of what traditionally has been called a *common name*.[6] The three important features of this notion are, first, that a vacuous common name, i.e., a common name that names nothing, has no extension, which is not the same as having an empty class as its extension. Thus, according to Russell, "there is no such thing as the null class, though there are null class-concepts," i.e., common-name concepts that have no extension.[7] Secondly, the extension of a common name that names just one thing (in the sense of an urelement) is just that one thing. In other words, unlike the singleton sets of set theory, which are not identical with their single member, the class that is the extension of a common name that names just one thing (urelement) is none other than that one thing. This fact is related to our third point, namely, that unlike sets, classes as the extensions of names are literally made up of their members so that when they have more than one member they are in some sense pluralities (*Vielheiten)*, or "plural objects," and not things that can themselves be members of classes. Thus, according to Russell, "though terms

[3]See, e.g., chapter 2 of Cocchiarella 1987.

[4]See, e.g., Simons 1982 for one proposed formulation of this notion. Simons's formulation is different from the one we give here. Simons doubts that there can be "an exact logic for the quantificatory uses" of common names, which is what the present system is based on. Also, whereas the present system relies on only one type of "objectual" variable (having both "atoms" and classes as many as values), Simons has three: one for "individuals," another for "pluralities," and a third for "neutrals." There are a number of other differences as well, but we will not go into them here.

[5]See Russell 1903, chapter VI. Russell's view in this book precedes his later no-classes doctrine.

[6]Strictly speaking, Russell distinguishes between a common name, e.g., 'man', and its plural form, 'men', and then takes the latter to denote the class as many of men (§67). We do not distinguish common names from their plural forms here, and we describe the class as many simply as the extension of the common name (and the concept it stands for).

[7]Russell 1903, §69.

[i.e., objects] may be said to belong to ... [a] class, the class [as a plurality] must not be treated as itself a single logical subject."[8] For example, "a plurality of terms is not the logical subject when a number is asserted of it: such propositions have not one subject, but many subjects, such as is conveyed by 'A and B' or 'A and B and C', or any other enumeration of definite terms."[9] Sometimes we also speak of a class as many in this sense—i.e., when it has two or more members—as a *group*, a practice we will adopt here as well. Thus, e.g., it would seem to be the notion of a group, rather than the more general notion of a class as many, that is the basis of the truth conditions of sentences with plural quantifier phrases, such as Peter Geach's example of 'Some critics admire only each other', which can be rephrased as 'There is a group of critics who admire only other members of the group'.[10]

We believe that this notion of a class, or of a group, can be usefully developed as part of the broader framework of conceptual realism that we have described elsewhere.[11] The core of this framework is a second-order predicate logic in which predicates can be nominalized and allowed to occur as abstract singular terms. Nominalized predicates do not denote classes in this framework, however, but intensional objects, sometimes also called "properties" and "relations," which are *object*-ifications of the truth conditions determined by the concepts (cognitive structures) that predicates stand for in their role as predicates and that underlie their rule-governed use in language. We will not be concerned with the core part of this framework here, however, but will restrict ourselves to an extension of the logic of names that

[8] Russell 1903, §70.

[9] Ibid., §§70 and 71. As a plurality, a class as many was understood by Russell to be an "*object* in a wider sense than *term*" or individual, though he recognized that there were "grave logical problems" associated with such a view (*op. cit.*, p. 55). In our reconstruction below, we take the notion of an object to be the same as that of an individual, though the *object*-ification of a class as many as the extension of a nominal concept is ontologically founded upon the conceptual mode of being of the concept, a fact that distinguishes classes as many from, e.g., ordinary physical objects. As a plurality, a class as many is sometimes also called a "plural object".

[10] This sentence, or a variant of it, is sometimes referred to as the Geach-Kaplan example, as, e.g., in Quine 1974, though where in Geach's or Kaplan's publications it occurs is left unspecified. Quine expressed the example as 'Some critics admire only one another' and claims that its analysis calls for "quantification over classes" (p. 111). (This reference was pointed out to me by A. Hazen.) Of course, Quine meant by a class only a single object (or "class as one" as Russell described them) and not a plurality or plural object such as Russell's notion of a class as many.

[11] See, e.g., Cocchiarella 1996 for a description of conceptual realism as a formal ontology.

is a central part of the theory of reference in conceptual realism. In particular, whereas names, both proper and common, and complex or simple, occur as parts of quantifier phrases in this theory, we propose that when these same names are nominalized and allowed to occur as singular terms on a par with individual variables, then, insofar as they can be taken as denoting anything at all, their denotata can be taken to be classes as many, i.e., classes in the sense indicated above.

1 Reference in Conceptual Realism

Reference in conceptual realism is a pragmatic notion in which both general and singular reference is given a unified account based on the category of names.[12] This means that the category of names in the logic of this theory contains symbolic counterparts of both proper and common names, i.e., both proper and common nouns, including common noun phrases (such as 'house that is brown', 'man who is wise', etc.). Common names are not restricted to sortal common names, i.e., common names that have identity criteria associated with their use (such as those just indicated), but include also such common names as 'event', 'physical object', 'thing', 'thing that is green', etc. Sortal common names are conceptually prior to common names in general, and the logic of sortals is not unimportant in conceptualism.[13] In fact, the concept of a thing (object, individual, etc.) is initially constructed on the basis of a thing of some sort or other.[14] But that is not an issue that will concern us here. We do restrict ourselves to the common names that are count nouns, however. Also, we refer to names in general, whether proper or common, complex or simple, as nominal expressions, and the concepts expressed by names as nominal concepts.

Names, both proper and common, are different from predicate expressions, as Peter Geach has pointed out, in that they can be "used outside the context of a sentence" in "simple acts of naming," which are semantically different kinds of acts from assertions.[15] In assertions, names are used in ref-

[12] See Cocchiarella 1998 for a more detailed description and defense of this theory of reference.

[13] For an account of the logic of sortals, see Cocchiarella 1977 and Freund 2001. We might also note that restricting nominal quantifiers to sortal concepts is particularly appropriate in A.I.

[14] See Sellars 1963, p. 253f, for precisely this sort of claim.

[15] See Geach 1980, p. 52. Naming is a pragmatic semiotical act that is also different

erential acts to refer to some, all, most, few, etc., of the things named, and in conceptualism that use is represented by quantifier phrases. An assertion of ‘Every man is mortal’, for example, is symbolized in our conceptualist theory as $(\forall x Man)Mortal(x)$, or more simply as $(\forall x A)F(x)$, where A is the common name symbol for ‘man’, F represents the verb phrase ‘is mortal’, and $(\forall x A)$ stands for a referential act to every man. Proper names occurring with an existential or universal quantifier represent forms of singular reference with and without existential presuppositions respectively. Thus, e.g., an assertion of ‘Socrates is mortal’ in which the referential act is with existential presupposition is symbolized as $(\exists x B)F(x)$, where B is a symbolic name for ‘Socrates’. An assertion of the same form, but without existential presupposition, is symbolized as $(\forall x B)F(x)$.

Referential acts can be “deactivated,” which is central to a conceptual-realist account of intensional verbs, but we will not be concerned with that part of the theory here.[16] In other words, the conceptualist logic of names that we will be concerned with here, and which we will call the *simple* logic of names, is really only a fragment of the full theory of reference of conceptual realism. It is of some interest in its own right, however, and in particular has been shown to provide an interpretation of Leśniewski's system of elementary ontology, which has also been called a logic of names.[17] We extend that result here by showing how the extensions of nominal concepts can be represented by nominalizing the names of this logic (whether proper or common, complex or simple), i.e., by transforming them into singular terms, the way ‘man’ can be transformed into ‘mankind’, which is different from the transformation of ‘human’ into ‘humanity’ (or of the complex predicate ‘is a man’, as represented by the λ-abstract $[\lambda x(\exists y Man)(x = y)]$, into the infinitive or gerundive phrases ‘to be a man’ and ‘being a man’).[18] These are two different types of nominalization, or transformations of nondenoting expressions into denoting expressions, i.e., into singular terms (expressions

from referring; and both naming and reference are different from denotation, which is not a pragmatic notion but a semantical one involved in characterizing the truth conditions of sentences containing nominalized expressions.

[16] See Cocchiarella 1998 for an account of the deactivation of referential concepts.

[17] See Cocchiarella 2001, where it is also shown that the simple logic of names is equivalent to second-order monadic predicate logic.

[18] See Sellars 1963 for another description of the difference between these two forms of nominalization. Russell, as already noted, did not use ‘mankind’ but the plural ‘men’ as a way of transforming ‘man’ into an expression that denotes the class as many of men, i.e., the class of men taken as a plurality.

that can be substituted for free individual variables); and the kind of entity that one denotes is different from the kind that the other denotes. Indeed, one denotes an extensional entity whereas the other denotes an intensional entity. Thus, according to Sellars, whereas 'S has (the property) whiteness' is the counterpart of 'S is white', 'S is a member of (the class) mankind' is the counterpart of 'S is a man'.[19] Similarly, according to Russell, whereas the counterpart of 'Socrates is human' is 'Socrates has (the property) humanity', the counterpart of 'Socrates is a man' is 'Socrates is one among men', where the plural 'men' in the last sentence denotes, according to Russell, the class as many of men, so that what the sentence says is that Socrates is a member of the class (as many) of men.[20] These different sentences express different, but equivalent, propositions according to Russell, and the second in each pair is derived from the first, just as they are in conceptual realism.

What a nominalized name denotes, accordingly, is different from what a nominalized predicate (adjective) denotes, and, as understood in conceptual realism, the extensions of nominal concepts are essentially what Russell referred to as "classes as many." There is one exception, of course; namely, that whereas Russell took a class as many to be the extension only of a common name, we take the extension (if any) of a proper name also to be a class as many, but one that is identical with its single member. Thus, when the proper name 'Socrates' is nominalized, i.e., occurs as a singular term and not as part of a quantifier expression representing a referential act, its extension is none other than Socrates himself, which of course is different from the property of being Socrates as represented by the λ-abstract $[\lambda x(\exists y Socrates)(x = y)]$. Similarly, whereas 'mankind' denotes the extension or class as many (collective totality) of humans, 'humanity', which is the nominalization of the predicate 'human', denotes the property of being human, which we take to be an intensional, abstract object, and not an extensional object.[21] As already noted, the full logic of conceptual realism contains a logic of nominalized predicates, and hence an account of abstract intensional objects (properties and relations-in-intension), but our goal here

[19] Sellars 1963, p. 253.

[20] Russell 1903, §68.

[21] The adjective 'human' is sometimes taken as a common name when what is really meant is 'human being' or 'thing that is human'. In this sense every predicate F can be transformed into a complex common name of the form 'thing that is F. Of course, the class of human beings, or humankind, which is identical with mankind, is no way to be confused with the property humanity

is not to consider the full logic but to develop only the logic of nominalized names, whether proper or common and complex or simple, as an account of classes as many, i.e., classes in the collective sense.

2 The Simple Logic of Names

The simple logic of names (which, as we said, is a fragment of the full logic of conceptual realism) can be described as a version of identity logic free of existential presuppositions regarding (so-called) singular terms (i.e., free individual variables and expressions that can be properly substituted for such) with both absolute and relative quantifier phrases (such as $(\forall x A)$ and $(\exists x A)$) binding individual variables as well as absolute quantifiers binding nominal variables. We use x, y, z, etc., with or without numerical subscripts, to refer to individual variables, and A, B, C, with or without numerical subscripts, to refer to name (nominal) variables.[22] Complex names are formed by adjoining (so-called "defining") relative clauses to names. We use '/', as in 'A/φ' to represent the adjunction of a formula φ to the name A (which may itself be complex). Thus, e.g., the quantifier phrase representing reference to a house that is brown would be symbolized as $(\exists x House/Brown(x))$. We take the universal quantifier, $\forall$, the (material) conditional sign, $\rightarrow$, the negation sign, $\neg$, and the identity sign, $=$, as primitive logical constants, and assume the others to be defined in the usual (abbreviatory) way. The absolute quantifier phrases $(\forall x)$ and $(\exists x)$ are read as 'Every*thing*' and 'Some*thing*' (or 'Every *object*' and 'Some *object*'), i.e., as implicitly containing the most general or ultimate common name 'thing' (which we take to be synonymous with 'object' and 'individual'). The quantifier phrases $(\forall A)$ and $(\exists A)$ are taken as referring to every or some nominal concept, respectively. Name constants are introduced in particular applications of the logic.[23]

Because complex names contain formulas, *names* and *formulas* are inductively defined simultaneously as follows[24]: (1) every name variable (or constant) is a *name*; (2) for all individual variables x, y, $(x = y)$ is a *formula*; and if φ, ψ are formulas, B is a name (complex or simple), and x and C

[22] We assume that there are denumerably many of both kinds of variables.

[23] The absolute quantifiers and the quantifiers for nominal concepts are in general understood to be relativized to a given universe of discourse in an applied form of the logic.

[24] We adopt the usual informal conventions for dropping parentheses and for sometimes using brackets instead of parentheses.

are an individual and a name variable respectively, then (3) $\neg\varphi$, (4) $(\varphi \rightarrow \psi)$, (5) $(\forall x)\varphi$, (6) $(\forall xB)\varphi$, and (7) $(\forall C)\varphi$ are *formulas*, and (8) B/φ and (9) $/\varphi$ are *names*. We assume the usual definitions of bondage and freedom for individual variables and of the proper substitution of one individual variable (or singular term such as a nominalized nominal expression when such are later added) for another in a formula, and similarly we assume the definitions of bondage and freedom of occurrences of nominal variables in formulas, and the proper substitution in a formula φ of a nominal variable (or constant) B for free occurrences of a nominal variable C. A complex name B/ξ is *free for* C in φ with respect to an individual variable x (as place holder) if (1) for each variable y such that $(\forall yC)$ occurs in φ and C is free at that occurrence, then y is free for x in B/ξ, and (2) no variable, nominal or individual, other than x that is free in B/ξ becomes bound when a free occurrence of C in φ is replaced by an occurrence of $B/\xi(y/x)$.[25] If a name B (complex or simple) is free for C in φ with respect to a variable x, then the proper substitution of B for C in φ with respect to x is represented by $\varphi(B[x]/C)$.

The axioms of the simple logic of names are those of the free logic of identity plus the axioms for nominal quantifiers:

Axiom 1: All tautologous formulas;

Axiom 2: $(\forall x)[\varphi \rightarrow \psi] \rightarrow [(\forall x)\varphi \rightarrow (\forall x)\psi]$;

Axiom 3: $(\forall C)[\varphi \rightarrow \psi] \rightarrow [(\forall C)\varphi \rightarrow (\forall C)\psi]$;

Axiom 4: $(\forall C)\varphi \rightarrow \varphi(B[x]/C)$, where B is free for C in φ with respect to x;

Axiom 5: $\chi \rightarrow (\forall C)\chi$, where C is not free in χ;

Axiom 6: $\chi \rightarrow (\forall x)\chi$, where x is not free in χ;

Axiom 7: $(\forall x)(\exists y)(x = y)$, where x, y are different variables;

Axiom 8: $x = x$;

Axiom 9: $x = y \rightarrow (\varphi \rightarrow \psi)$, where ψ is obtained from φ by replacing one or more free occurrences of x by free occurrences of y;

Axiom 10: $(\forall xA)\varphi \leftrightarrow (\forall x)[(\exists yA)(x = y) \rightarrow \varphi]$, where x, y are different

[25] The use of '/' in '$\xi(y/x)$' represents the result of properly substituting y for x in ξ, and should not be confused with the use of '/' to generate complex names.

variables;

Axiom 11: $(\forall xA/\psi)\varphi \leftrightarrow (\forall xA)[\psi \rightarrow \varphi]$.

We take *modus ponens* (MP) and universal generalization (UG) with respect to an individual or a nominal variable as inference rules. The rule of universal generalization for relative quantifiers,

$$(\mathrm{UG}_N) \qquad if \ \vdash \varphi, \ then \vdash (\forall xA)\varphi,$$

is derivable by (UG) from axiom 10. The usual laws for interchanging provably equivalent formulas and for rewriting bound variables are easily seen to be derivable as well. The universal instantiation law in free logic for individual variables,

$$(\exists/\mathrm{UI}) \qquad (\exists x)(x = y) \rightarrow [(\forall x)\varphi \rightarrow \varphi(y/x)],$$

where x, y are distinct variables and y is free for x in φ, is derivable by Leibniz's law (LL), i.e., axiom 9, (UG), axioms 2 and 6, and tautologous transformations. The theorems that are counterparts to axioms 10 and 11 for the existential quantifiers, namely,

T1: $\vdash (\exists xA)\varphi \leftrightarrow (\exists x)[(\exists yA)(x = y) \wedge \varphi]$,

and

T2: $\vdash (\exists xA/\psi)\varphi \leftrightarrow (\exists xA)[\psi \wedge \varphi]$

are also derivable by elementary transformations and the definitions for $\wedge$ and $\exists$. Also, because absolute quantifiers are viewed as implicitly containing the common name 'thing', we assume that axiom 11 has the following schema as a special instance.

T3: $\vdash (\forall x/\psi)\varphi \leftrightarrow (\forall x)[\psi \rightarrow \varphi]$.

The following are some obvious theorems that are easily seen to be provable.

T4: $(\forall x)\varphi \rightarrow (\forall xA)\varphi$.

T5: $(\forall xA)\varphi \rightarrow [(\exists zA)(y = z) \rightarrow \varphi(y/x)]$, where y is free for x in φ.

T6: $(\exists xA)(y = x) \rightarrow (\exists x)(y = x)$.

T7: $(\forall x)\varphi \leftrightarrow (\forall A)(\forall xA)\varphi$, where A is not free in φ.

Proof. The left-to-right direction follows by T4, (UG_N), quantifier laws. The right-to-left direction follows by first universally instantiating A to *thing identical to itself*, i.e., $/(x = x)$, so that by axiom 4 we have $\vdash (\forall A)(\forall xA)\varphi \rightarrow (\forall x/x = x)\varphi$, and, by T3, $\vdash (\forall x/x = x)\varphi \rightarrow (\forall x)[x = x \rightarrow \varphi]$, from which, by axiom 8, (UG), and axioms 2 and 1, $\vdash (\forall x/x = x)\varphi \rightarrow (\forall x)\varphi$; and from this the right-to-left direction of T7 follows. ■

3 On the Extensions of Names

We turn now to the "nominalization" of names, i.e., the transformation of names as parts of quantifier phrases into singular terms, by which we mean expressions that are substituends of free individual variables.[26] Where A is a name, proper or common, complex or simple, we take $[\hat{x}A]$ also to be a name, but one in which the variable x is bound. Thus, where A is a name and φ is a formula, $[\hat{x}A]$, $[\hat{x}A/\varphi]$, and $[\hat{x}/\varphi]$ are names in which all of the free occurrences of x in A and φ are bound. We read these expressions as 'the class (or group) of A', 'the class (or group) of A that are φ', and 'the class (or group) of *things* that are φ'. (We will speak of a class as many as a group when there is more than one thing in the class.)

We assume the simultaneous inductive definition of names is now extended to include names of this form as well. That is, (1) every name variable (or constant) is a *name*; (2) if a, b are either individual variables, name variables (or constants), or names of the form $[\hat{x}B]$, where x is an individual variable and B is a name (complex or simple), then $(a = b)$ is a *formula*; and if φ, ψ are formulas, B is a name (complex or simple), and x and C are an individual and a name variable, respectively, then (3) $\neg\varphi$, (4) $(\varphi \rightarrow \psi)$, (5) $(\forall x)\varphi$, (6) $(\forall xB)\varphi$, and (7) $(\forall C)\varphi$ are *formulas*, and (8) B/φ, (9) $/\varphi$, and (10) $[\hat{x}B]$ are *names*.

Note that by definition we now have formulas of the form $(\forall y[\hat{x}A])\varphi$, as well as those of the form $(\forall xA)\varphi$ and $(\forall yA(y/x))\varphi$ as in §2. We reduce the first to the last of these forms by adding the following axiom schema to those already listed in §2 (but now understood to apply to our extended notions of name and formula)[27]:

[26] It should be noted that in free logic being a substituend of free individual variables—i.e., being a *singular term*—is not the same as denoting a value of the bound individual variables. That is, in free logic some singular terms may denote nothing.

[27] Axiom 12 is for names as parts of quantifier phrases the analogue of λ-conversion for

Axiom 12: $(\forall y[\hat{x}A])\varphi \leftrightarrow (\forall y A(y/x))\varphi$, where y does not occur in A.

Given our understanding of the existential quantifier as defined in terms of negation and the universal quantifier, this means we also have the following as a theorem (where y is free for x in A):

T8: $\vdash (\exists y[\hat{x}A])\varphi \leftrightarrow (\exists y A(y/x))\varphi$.

Two other axioms about the occurrence of names as singular terms are:

Axiom 13: $(\exists A)(A = [\hat{x}B])$, where B is a name and A is a name variable that does not occur (free) in B; and

Axiom 14: $A = [\hat{x}A]$, where A is a simple name, i.e., a name variable or constant.

It is noteworthy that our earlier axiom 4 is now redundant and can be derived by the now extended version of Leibniz's law, (LL*), and axiom 13. That is, by (LL*), $\vdash C = [\hat{x}B] \rightarrow [\varphi \rightarrow \varphi([\hat{x}B]/C)]$, and therefore by (UG), axioms 3, 5, and tautologous transformations, $\vdash (\exists C)(C = [\hat{x}B]) \rightarrow [(\forall C)\varphi \rightarrow \varphi([\hat{x}B]/C)]$, and hence, by axiom 13,

T9: $\vdash (\forall C)\varphi \rightarrow \varphi([\hat{x}B]/C)$.

It should be noted that T9 is slightly stronger than axiom 4 in that it includes cases in which complex names occur as singular terms, i.e., where some (or all) of the occurrences of C in φ may be as singular terms, and hence where not all occurrences of $[\hat{x}B]$ in $\varphi([\hat{x}B]/C)$ can be replaced by B if B is a complex name of the form A/ψ or of the form $/\psi$. If C occurs in φ only as part of a quantifier phrase, then $\varphi([\hat{x}B]/C)$ is equivalent to $\varphi(B/C)$ by axiom 12.

We turn now to definitions of some of the concepts that are important in the logic of classes. Although we adopt the same symbols that are used in set theory to express membership, inclusion and proper inclusion, it should be kept in mind that the present notion of class is not that of set theory.

Definition 1 $x \in y \leftrightarrow (\exists A)[y = A \wedge (\exists z A)(x = z)]$.

Definition 2 $x \subseteq y \leftrightarrow (\forall z)[z \in x \rightarrow z \in y]$.

Definition 3 $x \subset y \leftrightarrow x \subseteq y \wedge y \nsubseteq x$.

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predicates.

Russell's paradox for classes does not lead to a contradiction within this system as described so far, it should be noted. Rather, what it shows is that the Russell class as many does not "exist" in the sense of being the value of a bound individual variable, which is not to say that the concept of the Russell class does not have its own conceptual mode of being as a value of the bound nominal variables. Indeed, as the following definition indicates, the nominal concept of the Russell class can be defined in purely logical terms.

Definition 4 $Rus = [\hat{x}/(\exists A)(x = A \wedge x \notin A)]$.

That the Russell class does not "exist" (as a value of the bound individual variables) is stated in the following theorem.[28]

T10: $\vdash \neg(\exists x)(x = Rus)$.

Proof. By axiom 13 and identity logic, $\vdash (\exists A)(Rus = A)$, and by definition 1, $\vdash Rus \in Rus \leftrightarrow (\exists A)[Rus = A \wedge (\exists x A)(x = Rus)]$, and therefore by Leibniz's law, a quantifier-confinement law and tautologous transformations, $\vdash Rus \in Rus \leftrightarrow (\exists x Rus)(x = Rus)$. But then, by definition of Rus and T8, $\vdash (\exists x Rus)(x = Rus) \leftrightarrow (\exists x/(\exists A)(x = A \wedge x \notin A))(x = Rus)$, and therefore, by T1, $\vdash (\exists x Rus)(x = Rus) \leftrightarrow (\exists x)[(\exists A)(x = A \wedge x \notin A) \wedge x = Rus]$, from which, by Leibniz's law, it follows that $\vdash (\exists x Rus)(x = Rus) \leftrightarrow (\exists x)[Rus \notin Rus \wedge x = Rus]$; and, accordingly, by quantifier-confinement laws, and tautologous transformations, $\vdash (\exists x)(x = Rus) \rightarrow (Rus \in Rus \leftrightarrow Rus \notin Rus)$, from which we conclude that $\vdash \neg(\exists x)(x = Rus)$. ■

What Russell's argument shows is that not every nominal concept has an extension that can be *object*ified (in the sense of being the value of an individual/object variable). The question arises, then, as to whether or not we can specify a necessary and sufficient condition for when a nominal concept has an extension that can be *object*ified, i.e., for when the extension of the concept can be proven to "exist" (as the value of a bound individual variable).[29] In fact, the answer is affirmative; i.e., unlike the situation in set

[28] That the Russell class as many does not exist as an object (i.e., as a value of the bound individual variables) is important to note because Boolos and others claim that "the objective view" of plural objects (such as classes as many) is refuted by Russell's paradox. (See, e.g., [Schein 1993], pages 5, 15, and 32-37.)

[29] By the *object*ification of the extension of a nominal concept we mean not only that the extension "exists" (in the sense of being a value of the bound individual variables), but also that the being of that extension as an object is founded upon that the (different mode of) being of the nominal concept whose extension it is. In particular, the extension

theory, such a condition can be specified for the notion of a class as many. An important part of this condition is Nelson Goodman's notion of an "atom," which, although it was intended for a strictly nominalistic framework, we can utilize for our purposes and define as follows.[30]

Definition 5 $Atom = [\hat{x}/\neg(\exists y)(y \subset x)]$.

This notion of an atom has nothing to do with physical atoms, of course. Rather, it corresponds in our present system approximately to the notion of an urelement in set theory. We say "approximately" because in this system atoms are identical with their singletons, and hence each atom will be a member of itself. This means that not only are ordinary physical objects atoms in this sense, but so are the propositions and intensional objects denoted by nominalized sentences and predicates in the fuller system of conceptual realism. The following axiom (where y does not occur in A) specifies when and only when a nominal concept A has an extension that can be *object*ified (as a value of the bound individual variables).

Axiom 15: $(\exists y)(y = [\hat{x}A]) \leftrightarrow (\exists xA)(x = x) \wedge (\forall xA)(\exists zAtom)(x = z)$.

Stated informally, axiom 15 says that the extension of a nominal concept A can be *object*ified (as a value of the bound individual variables) if, and only if, something is an A and every A is an atom.[31] An immediate consequence of this axiom, and of T8 and T1, is the following theorem schema, which stipulates exactly when an arbitrary condition φx has an extension that can be *object*ified.

T11: $\vdash (\exists y)(y = [\hat{x}/\varphi x]) \leftrightarrow (\exists x)\varphi x \wedge (\forall x/\varphi x)(\exists zAtom)(x = z)$.

Note that where φx is the impossible condition $(x \neq x)$, it follows from T11 that there can be no empty class, which, as already noted, is our first basic feature of the notion of a class as many. We define the empty-class concept

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could not exist (as an object) without the ontologically prior being (as a concept) of the nominal concept whose extension it is.

[30] See Goodman 1956 for Goodman's account of atoms in nominalism.

[31] That something is an A is perspicuously symbolized by $(\exists y)(\exists xA)(y = x)$. But because $(\exists xA)(x = x) \leftrightarrow (\exists y)(\exists xA)(y = x)$ is provable, we will use $(\exists xA)(x = x)$ as a shorter way of saying the same thing.

The referee for this paper has suggested that because A may have a dependency on x, the conjunct $(\forall xA)(\exists zAtom)(x = z)$ in Axiom 15 should be read as "no non-atom a is an $A(a/x)$".

as follows and then note that its extension, by T11, cannot "exist" (as a value of the bound individual variables), as well as that no *thing* can belong to it.

Definition 6 $\Lambda = [\hat{x}/(x \neq x)]$.

T12a: $\vdash \neg(\exists x)(x = \Lambda)$.

T12b: $\vdash \neg(\exists x)(x \in \Lambda$.

Finally, our last axiom concerns the second basic feature of classes as many; namely, that every atom (urelement) is identical with its singleton. In terms of a nominal concept A, the axiom stipulates that if at most one thing is an A and that whatever is an A is an atom, then whatever is an A is identical to the extension of A, which in that case is a singleton if in fact anything is an A. Where y does not occur in A, the axiom is as follows.

Axiom 16: $(\forall xA)(\forall yA)(x = y) \wedge (\forall xA)(\exists zAtom)(x = z) \rightarrow$
$(\forall yA)(y = [\hat{x}A])$.

A more explicit statement of the thesis that an atom is identical with its singleton is given in the following theorem.

T13: $\vdash (\exists zAtom)(x = z) \rightarrow x = [\hat{y}/(y = x)]$.

Proof. Where A be the nominal concept *thing-that-is-identical-to* x, i.e. $/(y = x)$, then, by axiom 11 and (LL*), $\vdash (\forall y/y = x)(\forall w/w = x)(y = w)$, and, similarly, $\vdash (\exists zAtom)(x = z) \rightarrow (\forall y/y = x)(\exists zAtom)(y = z)$. Therefore, by axiom 16, $\vdash (\exists zAtom)(x = z) \rightarrow (\forall y/y = x)(y = [\hat{y}/(y = x])$. But, by T6, $\vdash (\exists zAtom)(x = z) \rightarrow (\exists z)(z = x)$, and therefore by $(\exists/\text{UI})$, T3 and axiom 8, $\vdash (\exists zAtom)(x = z) \rightarrow x = [\hat{y}/(y = x]$. ■

By T13, it follows that every atom is identical with the extension of some nominal concept, e.g., the concept of being that atom. Of course, non-atoms are the extensions of nominal concepts as well (by the definitions of *Atom*, $\subset$, and $\in$), and hence anything whatsoever is the extension of a nominal concept.

T14: $\vdash (\exists zAtom)(x = z) \rightarrow (\exists A)(x = A)$.

T15: $\vdash \neg(\exists zAtom)(x = z) \rightarrow (\exists A)(x = A)$.

T16: $\vdash (\exists A)(x = A)$.

Note that if A is a proper name of an ordinary (physical) object (and hence an atom), then the antecedent of axiom 16 is true, and therefore, by axioms 16 and 14, $(\forall yA)(y = A)$. In other words, if A is a proper name of an atom, then $F(A) \leftrightarrow (\forall yA)F(y)$ is true[32], which in our conceptualist framework explains the role proper names have as "singular terms" (i.e., as substituends of free individual variables) in free logic. Of course, if A is a non-vacuous proper name of an ordinary object, then $(\exists yA)(y = A)$ is true, and hence $F(A) \leftrightarrow (\exists yA)F(y)$ as true as well.

A consequence of T13, the definition of $\in$, T8, and Leibniz's law is the thesis that every atom is a member of itself. A similar argument, but without T13, shows that every*thing* is a member of its singleton.

T17: $\vdash (\forall xAtom)(x \in x)$.

T18: $\vdash (\forall x)(x \in [\hat{z}/(z = x)])$.

Finally, we note that by definition of membership and Leibniz's law an object x can belong to the extension of a (nominal) concept A iff x is an A. From this it follows that only atoms can belong to an *object*ified class as many, and hence that classes as many that are not atoms are not themselves members of any (real) classes as many, which is our third basic feature of classes as many.

T19: $\vdash x \in A \leftrightarrow (\exists yA)(x = y)$.

T20a: $\vdash (\forall x)[z \in x \rightarrow (\exists wAtom)(z = w)]$.

T20b: $\vdash \neg(\exists wAtom)(z = w) \rightarrow \neg(\exists x)(z \in x)$.

Proof. By definition of $\in$, $\vdash z \in x \rightarrow (\exists A)[x = A \wedge (\exists wA)(z = w)]$, and therefore, by T6, $\vdash z \in x \rightarrow (\exists w)(z = w)$. By axiom 15, $\vdash (\exists y)(A = y) \rightarrow (\forall zA)(\exists wAtom)(z = w)$; and therefore, by axiom 10, T19, and (LL*), $\vdash (\exists y)(x = y) \wedge x = A \rightarrow (\forall z)[z \in A \rightarrow (\exists wAtom)(z = w)]$. But then, by quantifier-confinement laws, T16, (LL*), ($\exists$/UI) and elementary transformations, $\vdash (\exists y)(x = y) \rightarrow [z \in x \rightarrow (\exists wAtom)(z = w)]$. Therefore, by (UG) and axiom 7, $\vdash (\forall x)[z \in x \rightarrow (\exists wAtom)(z = w)]$, which is T20a. T20b then follows by a quantifier-confinement law and tautologous transformations. ∎

We should perhaps emphasize here that T20b does not preclude classes from being members of non-objectified classes as many, i.e., from classes as many that are not "real" in the sense of being the value of a bound

[32] That is, by (UG), axioms 2 and 6, T4, a quantifier-confinement law, and Leibniz's law,

$$(\forall yA)(y = A) \vdash F(A) \leftrightarrow (\forall yA)F(y).$$

individual variable. As we will see in §5 below, everything "real" in this sense is a member of the universal class, even though the universal class is not itself "real," i.e., cannot be objectified as a value of the bound individual variables. That everything "real" is a member of the universal class means only (by definition of $\in$) that everything real is self-identical.

4 Extensional Identity

The "nominalist's dictum," according to Nelson Goodman, is that "no two distinct things can have the same atoms."[33] Such a dictum, it would seem, should apply to classes as many (as traditionally understood), regardless whether or not a more comprehensive framework containing such classes is nominalistic or not. In fact, the dictum would be provable here if we were to assume an axiom of extensionality for classes; but in that case we would be committed to a strictly extensional framework even if it were not otherwise nominalistic. That is, nominal concepts that have the same extension at a given moment in a given possible world would then, by Leibniz's law, be necessarily equivalent, and therefore have the same extension at all times in every possible world. That is a consequence we do not want in our broader framework of conceptual realism, which is an intensional logic that, in keeping with our commonsense understanding of the world, contains a tense, modal, and epistemic logic, and therefore is not an extensional framework. Accordingly, we will not assume an axiom of extensionality here.[34]

Instead of strict identity, however, we can show that Goodman's nominalistic dictum holds for the weaker notion of "extensional identity," which is not in conflict with our wider intensional framework. Actually, even in an extensional framework where a tense, modal and epistemic logic are not assumed, one does not need to assume anything more than extensional identity. Of course, by things being extensionally identical, we mean in our present context only that whatever is in the one is in the other.

Definition 7 $(x =_{ex} y) \leftrightarrow (\forall z)[z \in x \leftrightarrow z \in y]$.

[33] Goodman 1956, p. 21.

[34] We could of course modify Leibniz's law so that it is provable only for extensional contexts and otherwise must be supplemented by certain special assumptions in order to apply to temporal and intensional contexts. In that case an extensionality axiom need not lead to undesirable consequences.

Restated in terms of extensional identity, Goodman's nominalistic dictum is that things that have the same atoms are extensionally the same. This version of the dictum, as already noted, is provable in our present system.

T21: $\vdash (\forall x)(\forall y)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow x =_{ex} y]$.

Proof. By T5, T20a, (UG), quantifier-confinement laws, and elementary transformations, $\vdash (\forall x)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow (\forall z)(z \in x \rightarrow z \in y)]$, and similarly $\vdash (\forall y)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow (z \in y \rightarrow z \in x)]$, from which T21 follows. ■

Note that by T13 and the definition of $\in$, whatever belongs to an atom is identical with that atom, and therefore atoms are extensionally identical if, and only if, they are identical *simpliciter*.

T22: $\vdash (\forall x Atom)[y \in x \rightarrow y = x]$.

T23: $\vdash (\forall x Atom)(\forall y Atom)[x =_{ex} y \leftrightarrow x = y]$.

Note also that by T21 (and other theorems) it follows that everything (real), whether it is an atom or not, has an atom in it.

T24: $\vdash (\forall x)(\exists z Atom)(z \in x)$.

Proof. By T5 and T17, $\vdash (\exists z Atom)(x = z) \rightarrow (\exists z Atom)(z \in x)$, and hence, by contraposition and the definition of $Atom$, $\vdash \neg(\exists z Atom)(z \in x) \rightarrow \neg(\exists z[\hat{x}\neg(\exists y)(y \subset x)])(x = z)$; and therefore, by axioms 12, 11 and elementary transformations, $\vdash \neg(\exists z Atom)(z \in x) \rightarrow (\forall z)[x = z \rightarrow (\exists y)(y \subset z)]$, from which, by (LL*) and a quantifier-confinement law, it follows that $\vdash \neg(\exists z Atom)(z \in x) \rightarrow [(\exists z)(x = z) \rightarrow (\exists y)(y \subset x)]$; and therefore, by (UG) and axioms 2 and 7, $\vdash (\forall x)[\neg(\exists z Atom)(z \in x) \rightarrow (\exists y)(y \subset x)]$. Now, by definition of $\subset$, $\vdash \neg(\exists z Atom)(z \in x) \wedge y \subset x \rightarrow \neg(\exists z Atom)(z \in y)$, and therefore $\vdash \neg(\exists z Atom)(z \in x) \wedge y \subset x \rightarrow (\forall z Atom)[z \in x \leftrightarrow z \in y]$, and, accordingly by (UG) and T21, $\vdash (\forall x)(\forall y)[\neg(\exists z Atom)(z \in x) \wedge y \subset x \rightarrow x =_{ex} y]$. But then, by definition of $\subset$, $\vdash (\forall x)(\forall y)[\neg(\exists z Atom)(z \in x) \rightarrow (y \subset x \rightarrow x \subseteq y \wedge x \not\subseteq y)]$; and therefore, by quantifier logic, $\vdash (\forall x)[\neg(\exists z Atom)(z \in x) \rightarrow \neg(\exists y)(y \subset x)]$. Together with the above result, this shows that $\vdash (\forall x)[\neg(\exists z Atom)(z \in x) \rightarrow (\exists y)(y \subset x) \wedge \neg(\exists y)(y \subset x)]$, from which T24 follows by quantifier logic,. ■

Another useful theorem is the following, which, together with T21, shows that every non-atom must have at least two atoms as elements. Of course, conversely, any real class (as many) that has at least two members cannot be an atom (because each of those members is properly contained in that class).

T25: $\vdash (\forall x)(\forall y)(y \subset x \rightarrow (\exists z Atom)[z \in x \wedge z \notin y])$.

Proof. By quantifier logic and definition of $\subset$, $\vdash y \subset x \rightarrow (\forall z Atom)(z \in y \rightarrow z \in x)$, and therefore, by (UG) and T21, $\vdash (\forall x)(\forall y)(y \subset x \rightarrow [(\forall z Atom)(z \in x \rightarrow z \in y) \rightarrow x =_{ex} y])$. But then, by definition of $\subset$ and $=_{ex}$, $\vdash (\forall x)(\forall y)(y \subset x \rightarrow [(\forall z Atom)(z \in x \rightarrow z \in y) \rightarrow x \subseteq y \wedge x \nsubseteq y])$, and hence $\vdash (\forall x)(\forall y)(y \subset x \rightarrow (\exists z Atom)[z \in x \wedge z \notin y])$. ■

T26: $\vdash (\forall x)[\neg(\exists y Atom)(x = y) \leftrightarrow (\exists z_1/z_1 \in x)(\exists z_2/z_2 \in x)(z_1 \neq z_2)]$.

Proof. By T25, $\vdash (\forall x)(\forall y)(y \subset x \rightarrow (\exists z_1 Atom)[z_1 \in x \wedge z_1 \notin y])$, and by T24 and ($\exists$/UI), $\vdash (\exists w)(y = w) \rightarrow (\exists z_2 Atom)(z_2 \in y)$. But, by (LL*) and definition of $\subset$, $\vdash y \subset x \wedge z_1 \notin y \wedge z_2 \in y \rightarrow z_2 \in x \wedge z_1 \neq z_2$, and therefore, by quantifier logic, $\vdash (\exists w)(y = w) \rightarrow (\forall x)[y \subset x \rightarrow (\exists z_1 Atom)(\exists z_2 Atom)(z_1 \neq z_2 \wedge z_1 \in x \wedge z_2 \in x)]$. Accordingly, by (UG), axiom 7, T1 and quantifier logic, $\vdash (\forall x)[(\exists y)(y \subset x) \rightarrow (\exists z_1 Atom/z_1 \in x)(\exists z_2 Atom/z_2 \in x))(z_1 \neq z_2)]$. But, by quantifier logic and definition of $Atom$, $\vdash (\forall x)[\neg(\exists y Atom)(x = y) \rightarrow (\exists y)(y \subset x)]$, from which the left-right-direction of T26 follows. The converse direction is of course trivial for the reason already noted. ■

Finally, let us note that there is an alternative to the axiom of extensionality that does not have the undesirable consequence that classes are strictly identical if as a matter only of contingent fact they are extensionally identical. In particular, instead of an *axiom* of extensionality, we can assume a *rule* to the effect that if classes are *provably* extensionally identical, then they are strictly identical. Of course, if it is provable that classes are extensionally identical, then it is also necessary that they are extensionally identical, and therefore, at least on this account, identical *simpliciter*. The rule in question can be stated as follows.

(Ext$\mapsto$Id) $\qquad$ *If* $\vdash (x =_{ex} y)$, *then* $\vdash x = y$.

One immediate consequence of (Ext$\mapsto$Id) is the derived rule that formulas that are provably equivalent have the same classes as many as their extensions.

(Ext$\mapsto$Id$_2$) $\qquad$ *If* $\vdash (\forall x)(\varphi \leftrightarrow \psi)$, *then* $\vdash [\hat{x}/\varphi] = [\hat{x}/\psi]$.

Proof. Assume $\vdash (\forall x)(\varphi \leftrightarrow \psi)$, and let y, z be distinct variables that are free for x in φ and ψ. Then, by definition of $\in$, $\vdash z \in [\hat{x}/\varphi] \leftrightarrow (\exists A)([\hat{x}/\varphi] = A \wedge (\exists y A)(z = y))$, and therefore, by (LL*) and T8, $\vdash z \in [\hat{x}/\varphi] \leftrightarrow (\exists y/\varphi(y/x))(z = y)$, and hence, by T2, $\vdash z \in [\hat{x}/\varphi] \leftrightarrow (\exists y)(\varphi(y/x) \wedge z = y)$;

and therefore, by assumption and interchange, $\vdash z \in [\hat{x}/\varphi] \leftrightarrow (\exists y)(\psi(y/x) \wedge z = y)$. But then, by the same reasoning, $\vdash z \in [\hat{x}/\psi] \leftrightarrow (\exists y)(\varphi(y/x) \wedge z = y)$, from which, by (UG), it follows that $\vdash [\hat{x}/\varphi] =_{ex} [\hat{x}/\psi]$, and therefore, by (Ext$\mapsto$Id), $\vdash [\hat{x}/\varphi] = [\hat{x}/\psi]$. ■

Two other consequences of (Ext$\mapsto$Id) are the strict identity of a class with the class of it members and the rewrite of bound variables for class expressions.[35]

T27a*: $\vdash x = [\hat{z}/(z \in x)]$.

T27b*: $\vdash [\hat{x}A] = [\hat{y}A(y/x)]$, where y does not occur in A.

Proof. By ($\exists$/UI), T2, and (LL*), $\vdash (\exists y)(z = y) \to [z \in x \to (\exists y/y \in x)(z = y)]$, and therefore, by T8 and T19, $\vdash (\exists y)(z = y) \to (z \in x \to z \in [\hat{z}/(z \in x)])$, and hence, by axiom 7, $\vdash (\forall z)(z \in x \leftrightarrow z \in [\hat{z}/(z \in x)])$. For the converse direction, by T19, (LL*), and T8, $\vdash z \in [\hat{z}/(z \in x)] \to (\exists y/y \in x)(z = y)$; and hence $\vdash z \in [\hat{z}/(z \in x)] \to z \in x$. Therefore, $\vdash x =_{ex} [\hat{z}/(z \in x)]$, and hence, by (Ext$\mapsto$Id), $\vdash x = [\hat{z}/(z \in x)]$. The proof that $\vdash [\hat{x}A] =_{ex} [\hat{y}A(y/x)]$ follows from the definition of $\in$ and the rewrite rule for relative quantifiers, and T27b then follows by (Ext$\mapsto$Id). ■

5 The Universal Class

We have seen that, unlike the situation in set theory, the empty class (as many) does not exist (as a value of the bound individual variables). But what about the universal class? In ZF set theory there is no universal set, but in NF and NFU there is. In our present theory, the situation is more complicated. For example, if nothing exists, then of course the universal class does not exist. But, in addition, because something exists only if an atom does, i.e., by T24 and ($\exists$/UI),

T28: $\vdash (\exists x)(x = x) \to (\exists x Atom)(x = x)$,

it follows that the universal class does not exist if there are no atoms—which is unlike the situation in set theory where classes exist whether or not there are any urelements. As it turns out, we can also show that the universal class does not exist if there are at least two atoms. If there is just one atom, however, the situation is more problematic. First, let us define

[35] In case we decide not to assume the rule (Ext$\mapsto$Id) in any given application of the logic, we will place a star (*) after every theorem for which it is assumed.

the universal class in the usual way, i.e., as the extension of the (nominal) concept of being a thing that is self-identical, and then note that whether or not the universal class can be objectified (as a value of the bound individual variables), nevertheless, everything "real" (in the sense of being the value of a bound individual variable) is in it. Despite appearances, it should be emphasized, all that T29 really says is that everything is a thing that is self-identical.

Definition 8 $V = [\hat{x}/(x = x)]$.

T29: $\vdash (\forall x)(x \in V)$.

Proof. By axiom 8, $\vdash (\forall x)(\exists y)(x = y) \leftrightarrow (\forall x)(\exists y)(y = y \wedge x = y)$, and therefore, by T2, $\vdash (\forall x)(\exists y)(x = y) \leftrightarrow (\forall x)(\exists y/y = y)(x = y)$, from which T29 follows by T8, T19 and the definition of V. ■

Now, by definition of $\in$, nothing can belong to the empty class, i.e., $x \notin \Lambda$, and therefore, by Leibniz's law, if anything at all exists, the universal class is not the empty class.

T30: $\vdash (\exists x)(x = x) \rightarrow V \neq \Lambda$.

But it does not follow that the universal class exists if anything does. Indeed, as already noted above, we can show that if there are at least two atoms, then the universal class does not exist. First, let us note that if something exists (and hence, by T28, there is an atom), then the class of atoms exists, i.e., then the class of atoms can be *object*ified (as a value of the bound individual variables).

T31: $\vdash (\exists x)(x = x) \rightarrow (\exists y)(y = Atom)$.

Proof. By axiom 15, $\vdash (\exists x Atom)(x = x) \wedge (\forall x Atom)(\exists y Atom)(x = y) \rightarrow (\exists y)(y = Atom)$, from which, by T28 and quantifier logic, T31 follows. ■

On the other hand, let us also note that if at least two atoms exist, then the class of atoms is not itself an atom.

T32: $\vdash (\exists x Atom)(\exists y Atom)(x \neq y) \rightarrow \neg(\exists z Atom)(z = Atom)$.

Proof. By definition of $\in$, T8, and elementary transformations, $\vdash x \neq y \rightarrow x \notin [\hat{z}/(z = y)] \wedge y \notin [\hat{z}/(z = x)]$, and therefore, by T13 and (LL*), $\vdash (\exists z Atom)(x = z) \wedge (\exists z Atom)(y = z) \wedge (x \neq y) \rightarrow x \notin y \wedge y \notin x$. By T20a, $\vdash (\exists z Atom)(x = z) \rightarrow x \subseteq Atom$, and, by T19, $\vdash (\exists z Atom)(y = z) \rightarrow y \in Atom$. Therefore, by definition of $\subset$, $\vdash (\exists z Atom)(x = z) \wedge$

$(\exists zAtom)(y = z) \wedge y \notin x \to x \subset Atom$, and hence $\vdash (\exists zAtom)(x = z) \wedge (\exists zAtom)(y = z) \wedge (x \neq y) \to x \subset Atom$. But, by definition of $Atom$, $\vdash (\forall x)(\forall y)[x \subset y \to \neg(\exists zAtom)(z = y)]$, and hence, by T31, T6, and $(\exists/\mathrm{UI})$, $\vdash (\exists zAtom)(x = z) \wedge x \subset Atom \to \neg(\exists zAtom)(z = Atom)$. Therefore, $\vdash (\exists xAtom)(\exists yAtom)(x \neq y) \to \neg(\exists zAtom)(z = Atom)$. ■

By means of T32, we can now show that if there are at least two atoms, then the universal class does not exist (as a value of the individual variables).

T33: $\vdash (\exists xAtom)(\exists yAtom)(x \neq y) \to \neg(\exists x)(x = V)$.

Proof. Note that by T20a and $(\exists/\mathrm{UI})$, $\vdash (\exists x)(x = V) \to (\forall x)[x \in V \to (\exists yAtom)(x = y)]$. But, by axiom 8, (UG), and axioms 2 and 6, $\vdash (\exists x)(x = V) \to (\exists x)(x = x)$, and hence, by T31 and $(\exists/\mathrm{UI})$, $\vdash (\exists x)(x = V) \to [Atom \in V \to (\exists yAtom)(y = Atom)]$. But, by T31, T29, and $(\exists/\mathrm{UI})$, $\vdash (\exists x)(x = V) \to Atom \in V$, and hence, $\vdash (\exists x)(x = V) \to (\exists yAtom)(y = Atom)$. Accordingly, by T32, $\vdash (\exists xAtom)(\exists yAtom)(x \neq y) \to \neg(\exists x)(x = V)$. ■

Finally, in regard to the question of whether or not the universal class exists if the universe consists of just one atom, note that if that were in fact the case, then, where A is a proper name of the one atom, the conjunction $(\exists zAtom)(z = A) \wedge (\forall zAtom)(z = A)$ would be true, and therefore the one atom A would be extensionally identical with the class of atoms, i.e., then, by T31 and T21, $(A =_{ex} Atom)$ would be true as well. Now, by T29 and T19, $(\forall zAtom)[z \in Atom \leftrightarrow z \in V]$ is provable, which, by T21 might suggest that $(Atom =_{ex} V)$ and hence that $(A =_{ex} V)$ are true as well. But in order for T21 to apply in this case we need to know that V exists, i.e., that $(\exists x)(x = V)$ is true. So, even if there were just one atom, we still could not conclude that the universal class is extensionally identical with that one atom. In any case, note that even if $(A =_{ex} V)$ were true, and we assumed an axiom of extensionality, so that $(A = V)$ were true as well, then, by Leibniz's law, the universal class would then both exist and be an atom. That would exclude the possibility that two or more atoms had existed in the past, or in general that there could have been more than one atom, a situation that should not be excluded on logical grounds alone.

6 Intersection, Union, and Complementation

Let us turn now to the Boolean operations of intersection, union and complementation for classes as many. We adopt the following standard definitions of each.

Definition 9 $x \cup y = [\hat{z}/z \in x \vee z \in y]$.

Definition 10 $x \cap y = [\hat{z}/(z \in x \wedge z \in y)]$.

Definition 11 $\bar{x} = [\hat{z}/z \notin x]$.

The following theorems regarding membership in the union and intersection of classes are consequences of T19 and T8. The proof of the theorem regarding membership in the complement of a class is slightly more involved.

T34: $\vdash (\forall z)(z \in x \cup y \leftrightarrow z \in x \vee z \in y)$.

T35: $\vdash (\forall z)(z \in x \cap y \leftrightarrow z \in x \wedge z \in y)$.

T36: $\vdash (\forall z)(z \in \bar{x} \leftrightarrow z \notin x)$.

Proof. By definition of $\in$, $\vdash z \in \bar{x} \leftrightarrow (\exists A)[\bar{x} = A \wedge (\exists yA)(z = y)]$, and therefore, by (LL*) and T8, $\vdash z \in \bar{x} \rightarrow (\exists y/y \notin x)(z = y)$, and hence, by T2 and (LL*), $\vdash z \in \bar{x} \rightarrow z \notin x$. For the converse direction, note that by T2 and (LL*), $\vdash (\exists y)(z = y) \wedge z \notin x \rightarrow (\exists y/y \notin x)(z = y)$, and therefore, by the definitions of $\in$ and $\bar{x}$, $\vdash (\exists y)(z = y) \rightarrow [z \notin x \rightarrow z \in \bar{x}]$, and hence by (UG) axioms 2 and 7, and elementary logic, $\vdash (\forall z)(z \notin x \rightarrow z \in \bar{x})$. ■

Two immediate consequences of T36 (together with T12b and T29) are that the empty class is extensionally identical with the complement of the universal class, and that the universal class is extensionally identical with the complement of the empty class; and, therefore, by (Ext$\mapsto$Id), it then follows that each extensional identity reduces to identity *simpliciter*.

T37a: $\vdash \Lambda =_{ex} \bar{V}$.

T37b*: $\vdash \Lambda = \bar{V}$.

T38a: $\vdash V =_{ex} \bar{\Lambda}$.

T38b*: $\vdash V = \bar{\Lambda}$.

In regard to the conditions for the existence of unions and intersections, we first prove a theorem that is useful in their respective proofs.

T39: $\vdash (\forall x)[(\exists z)(z \in x) \wedge (\forall z/z \in x)(\exists w Atom)(z = w)]$.

Proof. By T6, ($\exists$/UI), (LL*), and T17, $\vdash (\exists z Atom)(x = z) \rightarrow (\exists z)(z \in x)$, and, by T26, $\vdash (\forall x)[\neg(\exists z Atom)(x = z) \rightarrow (\exists z)(z \in x)]$; hence, $\vdash (\forall x)(\exists z)(z \in x)$. But then T39 follows by T20a and quantifier logic. ■

T40: $\vdash (\forall x)(\forall y)(\exists z)(z = x \cup y)$.

Proof. By T39 (twice), $\vdash (\forall x)[(\exists z)(z \in x) \wedge (\forall z/z \in x)(\exists w Atom)(z = w)]$ and $\vdash (\forall y)[(\exists z)(z \in y) \wedge (\forall z/z \in y)(\exists w Atom)(z = w)]$, and therefore, by quantifier logic, $\vdash (\forall x)(\forall y)[(\exists z)(z \in x \vee z \in y) \wedge (\forall z/z \in x \vee z \in y)(\exists w Atom)(z = w)]$. Accordingly, by T11, $\vdash (\forall x)(\forall y)(\exists z_1)(z_1 = [\hat{z}/(z \in x \vee z \in y)])$, from which T40 follows by definition of union. ■

The related theorem for intersection requires a qualification, because some intersections (e.g., of distinct atoms) are empty, and, the empty class (as many) does not exist. Clearly, the relevant qualification is that the classes being intersected have a member in common.

T41: $\vdash (\forall x)(\forall y)[(\exists z)(z \in x \wedge z \in y) \rightarrow (\exists z)(z = x \cap y)]$.

Proof. By T39 (twice) and elementary logic, $\vdash (\forall x)(\forall y)(\forall z/z \in x \wedge z \in y)(\exists w Atom)(z = w)]$, and therefore, by T11 and the definition of $\cap$, $\vdash (\forall x)(\forall y)[(\exists z/z \in x \wedge z \in y) \rightarrow (\exists z)(z = x \cap y)]$. ■

In regard to the existence of the complement of a class as many, we first note that if some atom is not in x, and therefore, by T36, is in $\bar{x}$, then the class as many of *atoms* in $\bar{x}$ exists, i.e., then $[\hat{z} Atom/(z \in \bar{x})]$ exists (as a value of the bound individual variables). This result cannot be shown for $\bar{x}$ alone, however, because, e.g., where $x = \Lambda$, then, by T38b, $\bar{x} = V$, in which case $\bar{x}$ does not exist, or at least not if there exist two or more atoms. Also, in that case $[\hat{z} Atom/(z \in \bar{x})] = Atom$, and therefore, by T28, $[\hat{z} Atom/(z \in \bar{x})]$ exists even though $\bar{x}$ does not.

T42: $\vdash (\exists z Atom)(z \notin x) \rightarrow (\exists y)(y = [\hat{z} Atom/(z \in \bar{x})])$.

Proof. By axiom 15, $\vdash (\exists z Atom)(z \in \bar{x}) \wedge (\forall z Atom/z \in \bar{x})(\exists w Atom)(z = w) \rightarrow (\exists y)(y = [\hat{z} Atom/(z \in \bar{x})])$; but, by axiom 11 and quantifier logic, $\vdash (\forall z Atom/z \in \bar{x})(\exists w Atom)(z = w)$, and therefore, by T36, $\vdash (\exists z Atom)(z \notin x) \rightarrow (\exists y)(y = [\hat{z} Atom/(z \in \bar{x})])$. ■

Note that we can show that an atom is in $[\hat{z} Atom/(z \in \bar{x})]$ iff it is in $\bar{x}$, but we cannot use this result (T43 below) to prove that $\bar{x}$ exists if $[\hat{z} Atom/(z \in \bar{x})]$ exists. In particular, we cannot use T21 to prove $[\hat{z} Atom/(z \in \bar{x})] =_{ex} \bar{x}$

unless we already know that both classes exist. And even given their extensional identity we still cannot prove their strict identity, because the extensionality rule (Ext$\mapsto$Id) requires that their extensional identity be unqualified. We give below the results that do hold here.

T43: $\vdash (\forall z Atom)(z \in [\hat{z}Atom/z \notin x] \leftrightarrow z \in \bar{x})$.

Proof. By T19, $\vdash (\exists y Atom/y \notin x)(z = y) \rightarrow z \in [\hat{y}Atom/y \notin x]$; and, by T19 and T36, $\vdash (\forall z Atom)(z \in [\hat{y}Atom/y \notin x] \rightarrow z \in \bar{x})$. For the converse direction, by T36 and T2, $\vdash (\forall z Atom)[z \in \bar{x} \rightarrow (\exists y Atom/y \notin x)(z = y)]$, and therefore, by T19, $\vdash (\forall z Atom)(z \in \bar{x} \rightarrow z \in [\hat{y}Atom/y \notin x])$. ■

T44: $\vdash (\exists y)(y = [\hat{z}Atom/z \notin x]) \wedge (\exists y)(y = \bar{x}) \rightarrow [\hat{z}Atom/z \notin x] =_{ex} \bar{x}$.

Proof. By T43, T21, and ($\exists$/UI). ■

7 Groups and Plural Reference

One of the uses of a logic of classes as many, as already noted, is that we can represent in a natural and intuitive way the truth conditions of sentences with plural quantifier phrases. We suggested, for example, that the sentence 'Some critics admire only each other' can also be read as 'There is group of critics who admire only each other'. Now, if by a group we meant only a class as many, then the logical form of this sentence would be as follows:[36]

$$(\exists x/x \subseteq [\hat{y}Critic])(\forall y/y \in x)(\forall z)[Admire(y, z) \rightarrow z \in x \wedge z \neq y].$$

The group of critics being posited in this formula cannot be empty, because there is no empty class as many. On the other hand, the formula does not exclude the possibility that the group of critics in question has only one member—and hence is identical with that one member—who admires no one, and who therefore vacuously satisfies the condition $(\forall y/y \in x)(\forall z)[Admire(y, z) \rightarrow z \in x \wedge z \neq y]$. Such a possibility does not seem to be part of the content of this sentence, i.e., one of the possible contexts in which it might be truthfully asserted; and that is because part of the content of a plural quantifier phrase is that the group referred to consists of more than one object. To remedy this we will assume that the common name 'group' can be used to refer only to a class as many that has a proper subclass, i.e.,

[36] This formulation is the counterpart in terms of classes as many of the version in terms of second-order monadic predicate logic in Boolos 1984, p. 432.

a class as many that is not an atom, and hence that every group has at least two elements (as in T45).

Definition 12 $Grp = [\hat{x}/(\exists y)(y \subset x)]$.

T45: $\vdash (\forall x Grp)(\exists z_1/z_1 \in x)(\exists z_2/z_2 \in x)(z_1 \neq z_2)$.

We can now symbolize the sentence 'Some critics admire only each other' in terms of a group of critics instead of just a class as many of critics as above; that is, assuming the above formulation was correct except for this difference, we can symbolize the sentence as follows:

$$(\exists x Grp/x \subseteq [\hat{y} Critic])(\forall y/y \in x)(\forall z)[Admire(y,z) \to z \in x \wedge z \neq y].$$

This formulation may also be somewhat problematic, however.[37] Consider, for example, a group x of three critics A, B, and C satisfying the following conditions: (1) A admires B but does not admire anyone else, (2) B admires A and C but does not admire anyone else, and (3) C admires A but does not admire anyone else. Then $x = [\hat{y}(y = A \vee y = B \vee y = C)] \subseteq [\hat{y} Critic]$ and $(\forall y/y \in x)(\forall z)[Admire(y,z) \to z \in x \wedge z \neq y]$ are true, and yet it seems counter-intuitive to claim that x is a group of critics who admire only each other. What else is needed, apparently, is the claim that distinct members of x admire each other; that is, that the following is a more appropriate analysis of the Geach sentence[38]:

$$\begin{aligned}(\exists x Grp/x \subseteq [\hat{y} Critic])[(\forall y/y \in x)(\forall z/z \in x)(y \neq z \to Admire(y,z)) \wedge \\ (\forall y/y \in x)(\forall z)(Admire(y,z) \to z \in x \wedge z \neq y)].\end{aligned}$$

Another referential expression that is used to refer to groups in this sense is the plural 'the', as in 'the inhabitants of London' and 'the sons of rich men'. These examples are from Russell's *Introduction to Mathematical Philosophy*, where Russell makes it clear that the references are to classes—even though

[37] I owe the following counter-example to Randall Holmes.

[38] In English, this says that there is a group of critics who admire each other *and* only each other. Whether or not this is the preferred reading of the original sentence depends on how one understands the use of 'only' in the context in which the sentence is used. That is, some (but not all) uses of 'only' intend a conjunction, as, e.g., in saying 'A loves only B' one means 'A loves B *and* only B'. But in 'Only the brave deserve the fair', we do not intend to mean 'All and only the brave deserve the fair'. The Geach sentence seems to be more like the latter.

he no longer accepts the notion of a class as "a primitive idea."[39] On our reading these expressions are to be taken as referring to the inhabitants of London as a group and similarly to the sons of rich men as a group. The plural 'the' can in this way be reduced to the singular 'the', i.e., to a definite description of a group.

Now, in conceptualism, the singular 'the' is represented by a quantifier (as are all determiners), e.g., $\exists_1$, where the truth conditions of an assertion of the form 'The A is F' are spelled out in essentially the Russellian manner (when the definite description is used with existential presupposition).[40] That is, although 'The A is F' is symbolized as $(\exists_1 xA)F(x)$, where $(\exists_1 xA)$ stands for a referential act (in which one purports to refer to a unique A) and $F(x)$ a predicable act whose mutual exercise in a speech or mental act results in an assertion of the type in question, the truth conditions of that act are perspicuously given in the following theorem of the background framework:[41]

$$\vdash (\exists_1 xA)F(x) \leftrightarrow (\exists xA)[(\forall yA)(y = x) \wedge F(x)].$$

Similarly, although 'The A that is F is G' is symbolized as $(\exists_1 xA/F(x))G(x)$, its truth conditions are given as follows:

$$\vdash (\exists_1 xA/F(x))G(x) \leftrightarrow (\exists xA)[(\forall yA)(F(y) \leftrightarrow y = x) \wedge G(x)].$$

Consider now the sentence 'The Greeks who fought at Thermopylae are heroes', which we take to be equivalent to 'The group of Greeks who fought at Thermopylae are heroes'. Using $F(x)$ for the verb phrases 'x fought at Thermopylae', we can symbolize the sentence as follows:

$$(\exists_1 xGrp/x = [\hat{x}Greek/F(x)])(\forall y/y \in x)(\exists zHero)(y = z).$$

[39] Russell 1919, p.181. Plural descriptions are also taken in this way in Simons 1982, p. 227.

[40] In conceptualism, it is sometimes necessary to distinguish the logical form representing the cognitive structure of a speech or mental act from the logical form that represents its truth conditions in the most perspicuous way—but where the connection between the two may be made explicit by meaning postulates or other principles. This, in fact, is the case with the use of definite descriptions. See Cocchiarella 1989, §6, for a fuller discussion of this distinction.

[41] Again, we are assuming that the definite description is being used with existential presuppositions. When the definite description is used without such presuppositions, it is symbolized as $(\forall_1 xA)F(x)$ and its truth conditions are as follows:

$$\vdash (\forall_1 xA)F(x) \leftrightarrow (\forall xA)[(\forall yA)(y = x) \rightarrow F(x)].$$

The truth conditions of this sentence, as noted above, amount to there being (now, at the time of the assertion) exactly one group of Greeks who fought at Thermopylae and every member of that group is a hero, which captures the intended content of the sentence in question.[42] We might also note that another standard formulation of the English sentence, namely, that the class as many of Greeks who fought at Thermopylae is contained within the class as many of heroes,

$$[\hat{x}Greek/F(x)] \subseteq [\hat{x}Hero],$$

is a consequence of the above formulation; and, in fact, if it is assumed that $[\hat{x}Greek/F(x)]$ has at least two members and that each of its members is an atom, then the two formulations are equivalent to one another.

Finally, there are also the kind of plural references we noted earlier from Russell's *Principles of Mathematics*, e.g., 'A and B and C are three (things)' or 'A and B are (at least) two of C's suitors', where A, B, and C are non-vacuous proper names. Russell did not include the parenthetical '(things)' and '(at least)' as we have, but these are really implicit in the examples.[43] The numerical phrases in question here are really quantifier phrases of the form $(\exists^3 x)$—or $(\exists^3 xThing)$, where the common name 'thing' is made explicit in the symbolism—and $(\exists^2_{\min} xD)$, where D is a common-name symbol for 'suitor of C'. The sentence form, 'At least two D are F', symbolized as $(\exists^2_{\min} xD)F(x)$, is equivalent to the following more perspicuous representation of its truth conditions,

$$\vdash (\exists^2_{\min} xD)F(x) \leftrightarrow (\exists xD)(\exists yD)[x \neq y \wedge F(x) \wedge F(y)],$$

and, of course, '(Exactly) two D are F' is equivalent to the following

$$\vdash (\exists^2 xD)F(x) \leftrightarrow (\exists xD)(\exists yD)[x \neq y \wedge F(x) \wedge F(y) \wedge (\forall zD)(z = x \vee z = y)].$$

The sentence 'A and B are (at least) two D', which we assume is equivalent to 'The group consisting of A and B has at least two members', can now be

[42] This reading does not exclude the possibility of there being another different group of Greeks in the future who will have fought at Thermopylae that might falsify the assertion.

[43] A common name, even if only the name 'thing', is always implicit in the use of a numerical quantifier phrase, as, e.g., in 'three books', 'at least two men', 'three things', etc. In regard to the parenthetical '(at least)', note that Russell's example is 'Brown and Jones are two of Miss Smith's suitors' (op. cit., §59), where it does not seem that Russell meant to imply that Miss Smith has only two suitors; otherwise the sentence should have been phrased slightly differently.

symbolized as

$$(\exists_1 xGrp/x = [\hat{y}/(y = A \vee y = B)])(\exists^2_{\min} zD)(z \in x),$$

which, given the assumption that A and B are non-vacuous proper names[44] of distinct atoms, is equivalent to[45]

$$(\exists^2_{\min} zD)(z \in [\hat{y}/(y = A \vee y = B)]).$$

The truth conditions of the sentence 'A and B and C are three (things)', which we assume is equivalent to 'The group consisting of A and B and C has three members', can similarly be perspicuously represented as

$$(\exists_1 xGrp/x = [\hat{y}/(y = A \vee y = B \vee y = C)])(\exists^3 z)(z \in x),$$

which, given the assumption that A, B, and C are non-vacuous proper names of distinct atoms, is equivalent to

$$(\exists^3 z)(z \in [\hat{y}/(y = A \vee y = B \vee y = C)]).$$

Now although these quantifier phrases are the basic kinds of ways by which we speak of different numbers of things, the natural numbers can also be expressed as predicable concepts of classes as many—and then, by nominalization, as abstract objects (concept-correlates) in their own right (i.e., as "properties" of classes as many).[46] The number three, for example,

[44]It should be kept in mind that non-vacuous proper names, when nominalized, behave in our present system just the way they do in so-called standard first-order predicate logic. That is why $(y = A \vee y = B)$ is well-formed and meaningful in our system.

[45]We note again that we are concerned here with a perspicuous representation of the truth conditions of the sentence in question, and not with a logical representation of it cognitive structure. The latter can be given as

$$(\exists xA \wedge \exists yB)[\lambda xy(\exists^2_{\min} zD)(z = x \vee z = y)](x, y),$$

where the conjunctive nominal phrase $(\exists xA \wedge \exists yB)$ is represented in the fuller framework of conceptual realism.

[46]We differ here from Russell's 1910 view of numbers as properties of properties, which was the way they were defined in [Cocchiarella 1989], §4. Note that on our present analysis,

$$0 = [\lambda x(\exists A)(x = A \wedge \neg(\exists yA)(y = y)],$$

i.e., zero is the "property" that nothing has (because there is no empty class), which is unlike our earlier analysis where zero is a property of properties that nothing has, i.e., a property that many things (properties) have.

can be defined (in the fuller framework of conceptual realism) as that concept under which fall all and only those classes as many that have three members, i.e.,

$$3 = [\lambda x(\exists A)(x = A \wedge (\exists^3 xA)(x = x)].$$

The sentence, 'The group consisting of A and B and C has three members', which given the assumption that A, B, and C are non-vacuous proper names of distinct atoms, can then also be symbolized as

$$3([\hat{y}/(y = A \vee y = B \vee y = C)]).$$

The logic of classes as many is more than a museum piece in logical and ontological analysis, we maintain. It provides an explication of a useful notion of classes other than that of sets (with or without a universal set, with or without an axiom of foundation, etc.), even if only for purposes of comparison regarding what it means for a class or set to have its being in its members. More importantly, it captures a notion that is basic to our commonsense understanding of groups of things, i.e., plural objects, and provides in this regard a natural semantics for many forms of plural reference. These kinds of references are a critical part of a proper understanding of our commonsense framework.

8 Appendix 1: A Set-Theoretic Semantics

A set-theoretical semantics can be constructed for the logic of classes as many as formalized here, and the system can be shown to be consistent with respect to that construction. We will forego the proofs in what follows and just sketch out the semantics for a "standard" model of the system without indices (possible worlds, moments of time, contexts of use, etc.), and therefore one in which the axiom of extensionality is valid. Extending this semantics to one that includes indices, and hence to one in which the axiom of extensionality is not valid, is unproblematic and can be done in the usual way.

By an (object) language we understand a (possibly empty) set of predicate and nominal constants. We will take '$\in$', '$\subseteq$', '$\subset$', and '$Atom$' defined in the logic of classes as primitive logical constants with their definitions as additional axioms of the logic. This means that the notion of a formula must be extended to include atomic formulas consisting of n-place predicate constants applied to n many singular terms, for $n \in \omega$.[47]

Definition 13 *L is a language iff L is a countable set of nominal constants (proper and common names) and predicate constants.*

By a set of "atoms" we understand a set that does not have the empty set as a member and no member of which has a member in common with that set. The idea is that "atoms" are to function as urelements. Thus, where D is any nonempty set, the set $\{\langle d, D\rangle : d \in D\}$ is a set of atoms even if D is not.

Definition 14 *D is a set of "atoms" iff D is a set such that $0 \notin D$ and for each $d \in D$, $d \cap D = 0$.*

A "standard" model for a language consists of a set, possibly empty, of "atoms", i.e., objects considered as urelements with respect to the various sets that make up the model, and an assignment of extensions to the predicate and nominal constants. The assignment to the constants is not drawn just from the set D of atoms, however, but from an extended set D^+ defined as follows.

Definition 15 *If D is a set (of atoms), then $D^+ = D \cup \{X \subseteq D : X \neq 0$ and for all $d \in D$, $X \neq \{d\}\}$.*

We define the denotation function with respect to a set D as follows.

Definition 16 *If D is a set (empty or otherwise), then the denotation function of D, in symbols, den_D, is that function with $D \cup \{X : X \subseteq D^+\}$ as domain and such that*
(1) for all $d \in D$, $den_D(d) = d$, and
(2) for all $X \subseteq D^+$,

$$den_D(X) = \begin{cases} d, \textit{ if } X = \{d\}, \textit{ for some (atom) } d \in D \\ X \textit{ otherwise} \end{cases} .$$

[47]We assume our metalanguage to be ZF set theory, and we take ω to be the set of natural numbers, where for each $n \in \omega$, n is the set of natural numbers less than n. Thus 0 is the empty set.

We now define the notion of a "standard" nominal model.

Definition 17 $\mathfrak{A}$ *is a nominal L-model iff L is a language and there are a set D, possibly empty, of "atoms" and a relation R such that*
(1) $\mathfrak{A} = \langle D, R \rangle$, *and*
(2) R is a function with L as domain and such that for each nominal constant $A \in L$, $R(A) \subseteq D^+$, *and for each* $n \in \omega$, *and each n-place predicate constant* $\pi \in L$, $R(\pi)$ *is a set of n-tuples drawn from* D^+.

Note that because $R(A) \subseteq D^+$, where A is a nominal constant, then names of single atoms are assigned singletons of those atoms, and not the atoms themselves. This is "corrected for" in the definition of the denotation function in a model, which, in the "standard" semantics, we identify with the denotation function of the domain of atoms in the model.

Definition 18 *If L is a language and* $\mathfrak{A} = \langle D, R \rangle$ *is a nominal L-model, then the denotation function in* $\mathfrak{A}$ *, in symbols,* $Den_{\mathsf{A}} = den_D$.

The following metatheorem indicates some of the useful features of the denotation function. Part (4) is the semantic analogue of the axiom of extensionality in the object language, which, as already noted, is valid in this semantics.

Metatheorem 1: If $\mathfrak{A} = \langle D, R \rangle$ is a nominal model, then
(1) if $d_1, d_2 \in D^+$, then $Den_{\mathsf{A}}(d_1) = Den_{\mathsf{A}}(d_2)$ iff $d_1 = d_2$;
(2) if $X, Y \subseteq D^+$ and $X, Y \notin D$, then $Den_{\mathsf{A}}(X) = Den_{\mathsf{A}}(Y)$ iff $X = Y$;
(3) if $d_1 \in D$, $d_2 \in D^+$ and for all $d_3 \in D^+$ [there is an $X \subseteq D^+$ such that $d_3 \in X$ & $Den_{\mathsf{A}}(d_2) = Den_{\mathsf{A}}(X)$ only if there is a $Y \subseteq D^+$ such that $d_3 \in Y$ & $Den_{\mathsf{A}}(d_1) = Den_{\mathsf{A}}(Y)$], then $d_1 = d_2$.
(4) if $d_1, d_2 \in D^+$: if for all $d_3 \in D^+$, (there is an $X \subseteq D^+$ such that $d_3 \in X$ & $Den_{\mathsf{A}}(d_1) = Den_{\mathsf{A}}(X)$ iff there is an $Y \subseteq D^+$ such that $d_3 \in Y$ & $Den_{\mathsf{A}}(d_2) = Den_{\mathsf{A}}(Y)$), then $d_1 = d_2$.

An assignment in a model of values to variables assigns objects in the union of the domain of atoms and the non-empty, nonsingleton subsets of that domain to the individual variables and subsets of the latter to the nominal variables.

Definition 19 *If* $\mathfrak{A} = \langle D, R \rangle$ *is a nominal L-model, then* $\mathfrak{a}$ *is an assignment (of values to variables) in* $\mathfrak{A}$ *iff* $\mathfrak{a}$ *is a function with the set of individual and*

nominal variables as domain and such that
(1) for each individual variable x, $\mathfrak{a}(x) \in D'$, for some $D' \supseteq D^+$, and
(2) for each nominal variable A, $\mathfrak{a}(A) \subseteq D^+$.

Thus, an assignment in a nominal model assigns values to the individual variables that are drawn from a set D' that contains D^+, i.e., $D^+ \subseteq D'$, and it assigns to the nominal variables subsets of D^+. The distinction between $D^+$and D' is required for the "free logic" aspect of the first-order part of the logic of classes as many; that is, D^+ is the set of values of bound individual variables and D' is the set of values of free individual variables.

We next inductively define the extension of a name or formula in a model.

Definition 20 *If L is a language, $\mathfrak{A} = \langle D, R\rangle$ is a nominal L-model, $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, and ξ is a name or formula of L, then the extension of ξ in $\mathfrak{A}$ relative to $\mathfrak{a}$ and an individual variable z (as place holder), in symbols $ext(\xi, \mathfrak{A}, \mathfrak{a}, z)$, is defined recursively as follows:*
(1) if ξ is a nominal variable or constant in L, then

$$ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \begin{cases} R(\xi), & \textit{if } \xi \textit{ is a constant in } L \\ \mathfrak{a}(\xi), & \textit{if } \xi \textit{ is a nominal or individual variable} \end{cases};$$

(2) if ξ is an identity formula $(a = b)$, where a, b are singular terms, i.e., either individual variables, nominal variables or constants in L, or names of L of the form $[\hat{x}B]$, then

$$ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \begin{cases} 1, & \textit{if } Den_{\mathsf{A}}(ext(a, \mathfrak{A}, \mathfrak{a}, z)) = Den_{\mathsf{A}}(ext(b, \mathfrak{A}, \mathfrak{a}, z)) \\ & 0 \textit{ otherwise} \end{cases};$$

(3) If ξ is $a \in b$, for some singular terms a, b of L, then $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 1$ if for some $X \subseteq D^+$, $ext(a, \mathfrak{A}, \mathfrak{a}, z) \in X$ & $Den_{\mathsf{A}}(b) = Den_{\mathsf{A}}(X)$; and otherwise $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 0$;

(4) If ξ is $a \subseteq b$, for some singular terms a, b of L, then $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 1$ if for all $d \in D^+$, there is an $X \subseteq D^+$ such that $d \in X$ & $Den_{\mathsf{A}}(a) = Den_{\mathsf{A}}(X)$ only if there is a $Y \subseteq D^+$ such that $d \in Y$ and $Den_{\mathsf{A}}(b) = Den_{\mathsf{A}}(Y)$; and otherwise $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 0$;

(5) If ξ is $a \subset b$, for some singular terms a, b of L, then $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 1$ if for all $d \in D^+$, there is an $X \subseteq D^+$ such that $d \in X$ & $Den_{\mathsf{A}}(a) = Den_{\mathsf{A}}(X)$

only if there is a $Y \subseteq D^+$ *such that* $d \in Y$ *and* $Den_{\mathsf{A}}(b) = Den_{\mathsf{A}}(Y)$*, and yet it is not the case that for all* $d \in D^+$*, there is an* $Y \subseteq D^+$ *such that* $d \in Y$ *&* $Den_{\mathsf{A}}(b) = Den_{\mathsf{A}}(Y)$*, only if there is an* $X \subseteq D^+$ *such that* $d \in X$ *and* $Den_{\mathsf{A}}(a) = Den_{\mathsf{A}}(X)$*;*

(6) if ξ *is* $\pi(a_1, ..., a_n)$*, for some* n*-place predicate constant in* L*, then* $ext_A(\xi, \mathfrak{A}, \mathfrak{a}, z) = 1$ *if* $\langle Den_{\mathsf{A}}(ext_A(a_1, \mathfrak{A}, \mathfrak{a}, z), ..., Den_{\mathsf{A}}(ext_A(a_n, \mathfrak{A}, \mathfrak{a}, z))\rangle \in R(\pi)$*; and otherwise* $ext_A(\xi, \mathfrak{A}, \mathfrak{a}, z) = 0$*;*

(7) if ξ *is Atom, then* $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = D$*;*

(8) if ξ *is* $\neg\varphi$*, for some formula* φ *of* L*, then*

$$ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \begin{cases} 1, \text{ if } ext(\varphi, \mathfrak{A}, \mathfrak{a}, z) = 0 \\ 0, \text{ otherwise} \end{cases} ;$$

(9) if ξ *is* $(\varphi \to \psi)$*, for some formulas* φ, ψ *of* L*, then*

$$ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \begin{cases} 0, \text{ if } ext(\varphi, \mathfrak{A}, \mathfrak{a}, z) = 1 \text{ and } ext(\psi, \mathfrak{A}, \mathfrak{a}, z) = 0 \\ 1, \text{ otherwise} \end{cases} ;$$

(10) if ξ *is* $(\forall x)\varphi$*, for some formula* φ *of* L *and individual variable* x*, then*

$$ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \begin{cases} 1 \text{ if for all } d \in D^+, \ ext(\varphi, \mathfrak{A}, \mathfrak{a}(d/x), z) = 1 \\ 0, \text{ otherwise} \end{cases} ;$$

(11) if ξ *is* $(\forall xB)\varphi$*, for some formula* φ *of* L*, individual variable* x*, and name* B*, then (note the change in place-holder)* $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 1$*, if for all* $d \in D^+ \cap ext(B, \mathfrak{A}, \mathfrak{a}(d/x), x)$*,* $ext(\varphi, \mathfrak{A}, \mathfrak{a}(d/x), x) = 1$*; and otherwise* $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = 0$;

(12) if ξ *is* $(\forall C\varphi)$*, for some formula* φ *of* L *and nominal variable* C*, then*

$$ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \begin{cases} 1, \text{ if for all } X \subseteq D^+, \ ext(\varphi, \mathfrak{A}, \mathfrak{a}(X/C), z) = 1 \\ 0, \text{ otherwise} \end{cases} ;$$

(13) if ξ *is* B/φ*, for some name* B *and formula* φ *of* L*, then* $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \{d \in D^+ : d \in ext(B, \mathfrak{A}, \mathfrak{a}(d/z), z) \ \& \ ext(\varphi, \mathfrak{A}, \mathfrak{a}(d/z), z) = 1\}$*;*

(14) if ξ *is* $/\varphi$*, for some formula* φ *of* L*, then* $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \{d \in D^+ : d \in ext(\varphi, \mathfrak{A}, \mathfrak{a}(d/z), z) = 1\}$; *and*

(15) if ξ *is* $[\hat{x}B]$*, for some individual variable* x *and name* B *of* L*, then* $ext(\xi, \mathfrak{A}, \mathfrak{a}, z) = \{d \in D^+ : d \in ext(B, \mathfrak{A}, \mathfrak{a}(d/x), z)\}$.

Note that in the definiens of clause 11 of this definition the place-holder variable z is replaced by the variable x. Also, although the place-holder in the definiens remains unchanged in clauses 13 and 14, the assignment is modified with respect to that place-holder. In clause 15, the place-holder is left unchanged and the assignment is modified with respect to the bound variable.

Definition 21 *If L is a language, φ is a formula of L, $\mathfrak{A}$ is a nominal L-model and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, then*
(1) a satisfies φ in $\mathfrak{A}$ iff $ext(\varphi, \mathfrak{A}, \mathfrak{a}, z) = 1$, for some individual variable z (as place-holder); and
(2) φ is true in $\mathfrak{A}$ iff every assignment in $\mathfrak{A}$ satisfies φ in $\mathfrak{A}$.

We define logical consequence and validity in the usual way.

Definition 22 *If L is a language and $\Gamma \cup \{\varphi\}$ is a set of formulas of L, then*
(1) φ is a logical consequence of Γ, in symbols, $\Gamma \models \varphi$, iff for every L-model $\mathfrak{A}$ and every assignment $\mathfrak{a}$ in $\mathfrak{A}$, if $\mathfrak{a}$ satisfies every formula in Γ in $\mathfrak{A}$, then $\mathfrak{a}$ satisfies φ in $\mathfrak{A}$; and
(2) φ is valid if it is a logical consequence of the empty set.

The following soundness theorem leads directly to a consistency proof for the logic of classes as many plus the axiom of extensionality.

Metatheorem 2: (Soundness) If φ is a theorem of the logic of classes as many plus the axiom of extensionality, then φ is valid with respect to the above semantics.

The consistency of the logic of classes as many (plus the axiom of extensionality) follows from the fact that $\{\langle n, \omega \rangle : n \in \omega\}$ is a set of atoms and therefore that $\langle \{\langle n, \omega \rangle : n \in \omega\}^+, 0 \rangle$ is a nominal model, and hence that every theorem is true in $\langle \{\langle n, \omega \rangle : n \in \omega\}^+, 0 \rangle$.

Metatheorem 3: The logic of classes as many with the axiom of extensionality added is consistent.

9 Appendix 2: Bell's System M[48]

The system M of classes as many in [Bell 2000] is different from the logic described here. M is designed to show how proper (or ultimate) classes, which do not belong to other classes, can be taken to be classes as many (though Bell takes all sets to be classes as many and redefines 'set' in terms of certain individuals identified as "labeled" classes). Unlike the logic described here, M is not designed to provide a semantics for plural references in natural language, and it is not clear how one might use it for that purpose. Nevertheless, Bell's system M can be translated into the logic of classes as many presented here with the result that, with the axiom of extensionality added to the latter, the translation of each axiom of M is a theorem of our present system (and hence that M is contained in the latter). M is a two-sorted first-order logic with capital letters, A, B, C, etc., for classes as many and lower-case letters x, y, z, etc., as individual variables. The logic of classes as many described here contains a two-sorted first-order (free) logic as a *proper* part, where the capital letters A, B, C, etc. for nominal concepts are nominalized and transformed into singular terms for classes as many. As primitive symbols, M also contains $\in$ for the membership relation, the identity sign (applied to classes terms or to individual terms separately), a labeling functor λ (applied to class terms), a co-labeling functor * (applied to individual terms), a monadic predicate I (applied to individual terms, where $I(a)$ is read 'a is an identifier'), a monadic predicate S (applied to class terms, where $S(A)$ is read 'A is a set'), and the abstraction operator, $\{x : \varphi(x)\}$ (read as 'the class defined by φ'). Our translation function τ identifies the capital and individual variables of M with the same letters in our logic of classes as many and interprets the labeling function λ as the nominalization of a nominal expression A, i.e., $\tau(\lambda A) = A$ (nominalized). Membership in M is identified with membership in the logic of classes as many and class abstracts are similarly identified with one another We extend τ so that, in addition to the correlation of the terms of M with singular terms of our logic of classes as many, each formula of M is translated into a formula of the logic of classes as many described here, with the translation of the co-labeling functor * given contextually, i.e., for formulas such as $\varphi(x^*)$ in which it occurs:

[48] After this paper was written and submitted for publication, another formulation for classes as many (but only in the context of set theory) was published in [Bell 2000]. This appendix was added to briefly explain the connection between that formulation and the one given here.

1. $\tau(x) = x$ and $\tau(A) = A$, for each individual variable x and nominal variable A;

2. $\tau(\lambda a) = \tau(a)$, where a is a class term of M (other than of the form b^*);

3. $\tau(\{x : \varphi(x)\}) = [\hat{x}/\tau(\varphi(x))]$;

4. $\tau(a \in b) = (\tau(a) \in \tau(b))$;

5. $\tau(a = b) = (a = b)$, where a, b are either both class terms of M or both terms for individuals;

6. $\tau(I(a)) = (\exists z)(\tau(a) = z)$, where z is the first individual variable not occurring in a;

7. $\tau(S(a)) = (\exists z)(\tau(a) = z)$;

8. $\tau(\varphi(a^*)) = (\exists A)[\tau(a) = A \wedge \tau(\varphi(A/a))]$, where a is a term of M for an individual and A is a class variable of M that is free for a in φ;

9. $\tau(\neg\varphi) = \neg\tau(\varphi)$; and $\tau(\varphi \to \psi) = [\tau(\varphi) \to \tau(\psi)]$;

10. $\tau(\forall x\varphi) = (\forall x)\tau(\varphi)$; and $\tau(\forall A\varphi) = (\forall A)\tau(\varphi)$

The translation of the axiom of extensionality of M is just a version of the axiom of extensionality in the present logic of classes as many. Also, the translation of each "labeling axiom" of M is an obvious theorem of our logic:

$$\tau(S(A) \leftrightarrow I(\lambda A)) = [(\exists z)(A = z) \leftrightarrow (\exists z)(A = x)],$$
$$\tau(I(x) \leftrightarrow S(x^*)) = [(\exists z)(x = z) \leftrightarrow (\exists A)[x = A \wedge (\exists z)(A = z)]],$$
$$\tau(S(B)) \to (\lambda B^*) = B) = [(\exists z)(B = z) \to (\exists A)[B = A \wedge A = B]],$$
$$\tau(I(x) \to \lambda(x^*) = x) = [(\exists z)(x = z) \to (\exists A)[x = A \wedge A = x]].$$

And finally the axiom of comprehension of M is also translated into a theorem of our logic:

$$\tau(y \in \{x : \varphi(x)\} \leftrightarrow \varphi(y/x)) = [(\exists z[\hat{x}/\tau(\varphi(x))])(y = z) \leftrightarrow \tau(\varphi(y/x))].$$

As noted, the labeling function of M that associates each class with a labeled individual corresponds to the nominalization transformation of our present logic of classes as many, and those cases where the labeled individuals are "sets" (as defined in M) correspond to those where a nominal concept is *object*ified (i.e., where its nominalization is a value of the bound individual variables).

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